Available online at www.sciencedirect.com
SciVerse ScienceDirect
Journal of Functional Analysis 262 (2012) 5110–5127
JOURNAL OF Functional Analysis
www.elsevier.com/locate/jfa

Local curvature-dimension condition implies measure-contraction property

Fabio Cavalletti [a,*], Karl-Theodor Sturm [b]

[a] *Hausdorff Center for Mathematics, Endenicher Allee 62, D-53115 Bonn, Germany*
[b] *Institut für Angewandte Mathematik Abt. Stochastische Analysis, Endenicher Allee 60, D-53115 Bonn, Germany*

Received 21 December 2011; accepted 21 February 2012
Available online 6 April 2012
Communicated by Cédric Villani

Abstract

We prove that for non-branching metric measure spaces the local curvature condition $\mathsf{CD}_{loc}(K,N)$ implies the global version of $\mathsf{MCP}(K,N)$. The curvature condition $\mathsf{CD}(K,N)$ introduced by the second author and also studied by Lott and Villani is the generalization to metric measure space of lower bounds on Ricci curvature together with upper bounds on the dimension. This paper is the following step of Bacher and Sturm (2010) [1] where it is shown that $\mathsf{CD}_{loc}(K,N)$ is equivalent to a global condition $\mathsf{CD}^*(K,N)$, slightly weaker than the usual $\mathsf{CD}(K,N)$. It is worth pointing out that our result implies sharp Bishop–Gromov volume growth inequality and sharp Poincaré inequality.

Keywords: Ricci curvature; Optimal transport; Metric geometry

1. Introduction

An important class of singular spaces is the one of metric measure spaces with generalized lower bounds on the Ricci curvature formulated in terms of optimal transportation. This class of spaces together with the condition on lower bounds on curvature have been introduced by the second author in [7,8] and independently by Lott and Villani in [3].

* Corresponding author.
E-mail addresses: fabio.cavalletti@hcm.uni-bonn.de (F. Cavalletti), sturm@uni-bonn.de (K.T. Sturm).

0022-1236/$ – see front matter
http://dx.doi.org/10.1016/j.jfa.2012.02.015

The condition called *curvature-dimension condition* $\mathsf{CD}(K,N)$ depends on two parameters K and N, playing the role of a curvature and dimension bound, respectively. We recall two important properties of the condition $\mathsf{CD}(K,N)$:

- the curvature-dimension condition is stable under convergence of metric measure spaces with respect to the L^2-transportation distance $\mathbb{D}$ introduced in [7];
- a complete Riemannian manifold satisfies $\mathsf{CD}(K,N)$ if and only if its Ricci curvature is bounded from below by K and its dimension from above by N.

Moreover a broad variety of geometric and functional analytic properties can be deduced from the curvature-dimension condition $\mathsf{CD}(K,N)$: the Brunn–Minkowski inequality, the Bishop–Gromov volume comparison theorem, the Bonnet–Myers theorem, the doubling property and local Poincaré inequalities on balls. All these listed results are quantitative results (volume of intermediate points, volume growth, upper bound on the diameter and so on) depending on K, N.

A variant of $\mathsf{CD}(K,N)$ is the *measure-contraction property*, $\mathsf{MCP}(K,N)$, introduced in [5] and [8]. In the setting of non-branching metric measure spaces it is proven that condition $\mathsf{CD}(K,N)$ implies $\mathsf{MCP}(K,N)$. While $\mathsf{CD}(K,N)$ is a condition on the optimal transport between any pair of absolutely continuous (w.r.t. m) probability measure on M, $\mathsf{MCP}(K,N)$ is a condition on the optimal transport between Dirac masses and the uniform distribution m on M. Nevertheless a great part of the geometric and functional analytic properties verified by spaces satisfying the condition $\mathsf{CD}(K,N)$ are also verified by spaces satisfying the $\mathsf{MCP}(K,N)$:

- generalized Bishop–Gromov volume growth inequality;
- doubling property;
- a bound on the Hausdorff dimension;
- generalized Bonnet–Myers theorem.

Again these results are in a quantitative form depending on K, N. For a complete list of analytic consequences of the measure-contraction property see [8].

Among the relevant questions on $\mathsf{CD}(K,N)$ that are still open, we are interested in studying the following one: can we say that a metric measure space (M,d,m) satisfies $\mathsf{CD}(K,N)$ provided $\mathsf{CD}(K,N)$ holds true locally on a family of sets M_i covering M?

In other words it is still not known whether $\mathsf{CD}(K,N)$ verifies the globalization property (or the local-to-global property).

A partial answer to this problem is contained in the work by Bacher and the second author [1]: they proved that if a metric measure space (M,d,m) verifies the local curvature-dimension condition $\mathsf{CD}_{loc}(K,N)$ then it verifies the global reduced curvature-dimension condition $\mathsf{CD}^*(K,N)$. The latter is strictly weaker than $\mathsf{CD}(K,N)$ and a converse implication can be obtained only changing the value of the lower bound on the curvature: condition $\mathsf{CD}^*(K,N)$ implies $\mathsf{CD}(K^*,N)$ where $K^* = K(N-1)/N$. Therefore $\mathsf{CD}^*(K,N)$ gives worse geometric and analytic information than $\mathsf{CD}(K,N)$.

In this paper we prove that if (M,d,m) is a non-branching metric measure space that verifies $\mathsf{CD}_{loc}(K,N)$ then (M,d,m) verifies $\mathsf{MCP}(K,N)$.

Hence our result implies that from the local condition $\mathsf{CD}_{loc}(K,N)$ one can obtain all the global geometric and functional analytic consequences implied by $\mathsf{MCP}(K,N)$ and therefore the geometric and functional analytic consequences are obtained in the sharp quantitative version.

We now present our approach to the problem.

As already pointed out, the curvature-dimension condition $\mathsf{CD}(K,N)$ prescribes how the volume of a given set is affected by curvature when it is moved via optimal transportation. Condition $\mathsf{CD}(K,N)$ imposes that the distortion is ruled by the coefficient $\tau_{K,N}^{(t)}(\theta)$ depending on the curvature K, on the dimension N, on the time of the evolution t and on the point θ.

The main feature of the coefficient $\tau_{K,N}^{(t)}(\theta)$ is that it is obtained mixing two different information on how the volume should evolve: an $(N-1)$-dimensional distortion depending on the curvature K by and a 1-dimensional evolution that doesn't feel the curvature. To be more precise

$$\tau_{K,N}^{(t)}(\theta) = t^{1/N}\sigma_{K,N-1}^{(t)}(\theta)^{(N-1)/N},$$

where $\sigma_{K,N-1}^{(t)}(\theta)^{(N-1)/N}$ contains the information on the $(N-1)$-dimensional volume distortion and the evolution in the remaining direction is ruled just by $t^{1/N}$. This is a clear similarity with the Riemannian case.

Our aim is, starting from $\mathsf{CD}_{loc}(K,N)$, to isolate a local $(N-1)$-dimensional condition ruled by the coefficient $\sigma_{K,N-1}^{(t)}(\theta)$ and then, using the easier structure of $\sigma_{K,N-1}^{(t)}(\theta)$, obtain a global $(N-1)$-dimensional condition with coefficient $\sigma_{K,N-1}^{(t)}(\theta)$. At that point, using Hölder inequality and the linear behavior of the other direction, it is not difficult to pass from the $(N-1)$-dimensional version to the full-dimensional version with coefficient $\tau_{K,N}^{(t)}(\theta)$.

However to detect a local $(N-1)$-dimensional condition it is necessary to decompose the whole evolution into a family of $(N-1)$-dimensional evolutions. Considering the optimal transport between a Dirac mass in o and the uniform distribution m, the family of spheres around x_0 immediately provides the correct $(N-1)$-dimensional evolutions. This motivates why we obtain $\mathsf{MCP}(K,N)$ and not $\mathsf{CD}(K,N)$.

We state the main result of this paper.

Theorem 1.1. *(Theorem 6.2.) Let (M,d,m) be a non-branching metric measure space. Assume that (M,d,m) satisfies $\mathsf{CD}_{loc}(K,N)$. Then (M,d,m) satisfies $\mathsf{MCP}(K,N)$.*

We end this paper with an outlook on the most general case we can address using the approach described so far.

2. Preliminaries

Let (M,d) be a metric space. The length $\mathsf{L}(\gamma)$ of a continuous curve $\gamma:[0,1]\to M$ is defined as

$$\mathsf{L}(\gamma) := \sup \sum_{k=1}^{n} d\big(\gamma(t_{k-1}),\gamma(t_k)\big)$$

where the supremum runs over $n\in\mathbb{N}$ and over all partitions $0=t_0<t_1<\cdots<t_n=1$. Clearly $\mathsf{L}(\gamma)\geqslant d(\gamma(0),\gamma(1))$. The curve is called *geodesic* if and only if $\mathsf{L}(\gamma)=d(\gamma(0),\gamma(1))$. In this case we always assume that γ has constant speed, i.e. $\mathsf{L}(\gamma\llcorner_{[s,t]}) = |s-t|\mathsf{L}(\gamma) = |s-t|d(\gamma(0),\gamma(1))$ for every $0\leqslant s\leqslant t\leqslant 1$.

With $\mathcal{G}(M)$ we denote the space of geodesic $\gamma:[0,1]\to M$ in M, regarded as subset of $Lip([0,1],M)$ of Lipschitz functions equipped with the topology of uniform convergence.

(M,d) is said a *length space* if and only if for all $x,y\in M$,

$$d(x,y)=\inf \mathsf{L}(\gamma)$$

where the infimum runs over all continuous curves connecting x to y. It is said to be a *geodesic space* if and only if every $x,y\in M$ are connected by a geodesic.

Definition 2.1. A geodesic space (M,d) is *non-branching* if and only if for all $r\geqslant 0$ and $x,y\in M$ such that $d(x,y)=r/2$ the set

$$\{z\in M:\ d(x,z)=r\}\cap\{z\in M:\ d(y,z)=r/2\}$$

is a singleton.

A *metric measure space* will always be a triple (M,d,m) where (M,d) is a complete separable metric space and m is a locally finite measure (i.e. $m(B_r(x))<\infty$ for all $x\in M$ and all sufficiently small $r>0$) on M equipped with its Borel σ-algebra. We exclude the case $m(M)=0$. A *non-branching* metric measure space will be a metric measure space (M,d,m) such that (M,d) is a non-branching geodesic space. Throughout the following we will use the notation $B_p(z)=\{x:\ d(x,z)<p\}$.

2.1. Geometry of metric measure spaces

$\mathcal{P}_2(M,d)$ denotes the L^2-Wasserstein space of probability measures on M and d_W the corresponding L^2-Wasserstein distance. The subspace of m-absolutely continuous measures is denoted by $P_2(M,d,m)$. A point z will be called t-intermediate point of points x and y if $d(x,z)=td(x,y)$ and $d(z,y)=(1-t)d(x,y)$.

The following are well-known results in optimal transportation and are valid for general metric measure spaces.

Lemma 2.2. *Let (M,d,m) be a metric measure space. For each geodesic $\Gamma:[0,1]\to\mathcal{P}_2(M)$ there exists a probability measure Ξ on $\mathcal{G}(M)$ such that*

- $e_{t\sharp}\Xi=\Gamma(t)$ *for all $t\in[0,1]$;*
- *for each pair (s,t) the transference plan $(\gamma_s,\gamma_t)_\sharp\Xi$ is an optimal coupling.*

The curvature-dimension condition $\mathsf{CD}(K,N)$ is defined in terms of convexity properties of the lower semi-continuous Rényi entropy functional

$$\mathcal{S}_N(\mu|m):=-\int_M \varrho^{-1/N}(x)\,\mu(dx) \tag{2.1}$$

on $P_2(M,d)$ where ϱ denotes the density of the absolutely continuous part μ^c in the Lebesgue decomposition $\mu=\mu^c+\mu^s=\varrho m+\mu^s$.

Given two numbers $K, N \in \mathbb{R}$ with $N \geqslant 1$, we put for $(t,\theta) \in [0,1] \times \mathbb{R}_+$,

$$\tau_{K,N}^{(t)}(\theta) := \begin{cases} \infty, & \text{if } K\theta^2 \geqslant (N-1)\pi^2, \\ t^{1/N}\big(\frac{\sin(t\theta\sqrt{K/(N-1)})}{\sin(\theta\sqrt{K/(N-1)})}\big)^{1-1/N} & \text{if } K\theta^2 \leqslant (N-1)\pi^2, \\ t & \text{if } K\theta^2 < 0 \text{ or} \\ & \text{if } K\theta^2 = 0 \text{ and } N = 1, \\ t^{1/N}\big(\frac{\sinh(t\theta\sqrt{-K/(N-1)})}{\sinh(\theta\sqrt{-K/(N-1)})}\big)^{1-1/N} & \text{if } K\theta^2 \leqslant 0 \text{ and } N > 1. \end{cases} \tag{2.2}$$

That is, $\tau_{K,N}^{(t)}(\theta) := t^{1/N}\sigma_{K,N-1}^{(t)}(\theta)^{(N-1)/N}$ where

$$\sigma_{K,N}^{(t)}(\theta) = \frac{\sin(t\theta\sqrt{K/N})}{\sin(\theta\sqrt{K/N})},$$

if $0 < K\theta^2 < N\pi^2$ and with appropriate interpretation otherwise. Moreover we put

$$\varsigma_{K,N}^{(t)}(\theta) := \tau_{K,N}^{(t)}(\theta)^N.$$

The coefficients $\tau_{K,N}^{(t)}(\theta)$, $\sigma_{K,N}^{(t)}(\theta)$ and $\varsigma_{K,N}^{(t)}(\theta)$ are all volume distortion coefficients depending on the curvature K and on the dimension N.

Definition 2.3 *(Curvature-dimension condition).* Let two numbers $K, N \in \mathbb{R}$ with $N \geqslant 1$ be given. We say that (M,d,m) satisfies the curvature-dimension condition – denoted by $\mathsf{CD}(K,N)$ – if and only if for each pair $\nu_0, \nu_1 \in \mathcal{P}_2(M,d,m)$ there exists an optimal coupling π of $\nu_0 = \varrho_0 m$ and $\nu_1 = \varrho_1 m$, and a geodesic $\Gamma : [0,1] \to \mathcal{P}_2(M,d,m)$ connecting ν_0 and ν_1 with

$$\mathcal{S}_{N'}\big(\Gamma(t)|m\big) \leqslant -\int_{M\times M} \big[\tau_{K,N'}^{(1-t)}\big(d(x_0,x_1)\big)\varrho_0^{-1/N'}(x_0) + \tau_{K,N'}^{(t)}\big(d(x_0,x_1)\big)\varrho_1^{-1/N'}(x_1)\big]\,\pi(dx_0\,dx_1), \tag{2.3}$$

for all $t \in [0,1]$ and all $N' \geqslant N$.

We recall also the definition of the reduced curvature-dimension condition $\mathsf{CD}^*(K,N)$ introduced in [1] as well as the definition of $\mathsf{CD}_{loc}(K,N)$.

Definition 2.4 *(Reduced curvature-dimension condition).* Let two numbers $K, N \in \mathbb{R}$ with $N \geqslant 1$ be given. We say that (M,d,m) satisfies the reduced curvature-dimension condition – denoted by $\mathsf{CD}^*(K,N)$ – if and only if for each pair $\nu_0, \nu_1 \in \mathcal{P}_2(M,d,m)$ there exists an optimal coupling π of $\nu_0 = \varrho_0 m$ and $\nu_1 = \varrho_1 m$, and a geodesic $\Gamma : [0,1] \to \mathcal{P}_2(M,d,m)$ connecting ν_0 and ν_1 such that (2.3) holds true for all $t \in [0,1]$ and all $N' \geqslant N$ with the coefficients $\tau_{K,N}^{(t)}(d(x_0,x_1))$ and $\tau_{K,N}^{(1-t)}(d(x_0,x_1))$ replaced by $\sigma_{K,N}^{(t)}(d(x_0,x_1))$ and $\sigma_{K,N}^{(1-t)}(d(x_0,x_1))$, respectively.

Definition 2.5 *(Local curvature-dimension condition).* Let two numbers $K, N \in \mathbb{R}$ with $N \geqslant 1$ be given. We say that (M,d,m) satisfies the curvature-dimension condition locally – denoted by $\mathsf{CD}_{loc}(K,N)$ – if and only if each point $x \in M$ has a neighborhood $M(x)$ such that for each pair

$\nu_0, \nu_1 \in \mathcal{P}_2(M,d,m)$ supported in $M(x)$ there exists an optimal coupling π of $\nu_0 = \varrho_0 m$ and $\nu_1 = \varrho_1 m$, and a geodesic $\Gamma : [0,1] \to \mathcal{P}_2(M,d,m)$ connecting ν_0 and ν_1 with

$$\mathcal{S}_{N'}\big(\Gamma(t)|m\big) \leqslant -\int_{M\times M} \big[\tau_{K,N'}^{(1-t)}\big(d(x_0,x_1)\big)\varrho_0^{-1/N'}(x_0) + \tau_{K,N'}^{(t)}\big(d(x_0,x_1)\big)\varrho_1^{-1/N'}(x_1)\big]\,\pi(dx_0\,dx_1), \tag{2.4}$$

for all $t \in [0,1]$ and all $N' \geqslant N$.

Notice that the geodesic Γ of the above definition can exit from the neighborhood $M(x)$.

As already emphasized in the introduction, in [1] it is proved that $\mathsf{CD}_{loc}(K,N)$ implies $\mathsf{CD}^*(K,N)$.

If a non-branching metric measure space (M,d,m) satisfies $\mathsf{CD}(K,N)$ then the uniqueness of geodesics can be proven. The next result is taken from [8].

Lemma 2.6. *Assume that (M,d,m) is non-branching and satisfies $\mathsf{CD}(K,N)$ for some pair (K,N). Then for every $x \in \operatorname{supp}[m]$ and m-a.e. $y \in M$ (with the exceptional set depending on x) there exists a unique geodesic between x and y.*

Moreover, there exists a measurable map $\gamma : M^2 \to \mathcal{G}(M)$ such that for $m \otimes m$-a.e. $(x,y) \in M^2$ the curve $t \mapsto \gamma_t(x,y)$ is the unique geodesic connecting x and y.

In the setting of non-branching metric measure space $\mathsf{CD}(K,N)$ has an equivalent pointwise formulation: (M,d,m) satisfies $\mathsf{CD}(K,N)$ if and only if for each pair $\nu_0, \nu_1 \in \mathcal{P}_2(M,d,m)$ and each optimal coupling π of them

$$\varrho_t\big(\gamma_t(x_0,x_1)\big) \leqslant \big[\tau_{K,N'}^{(1-t)}\big(d(x_0,x_1)\big)\varrho_0^{-1/N'}(x_0) + \tau_{K,N'}^{(t)}\big(d(x_0,x_1)\big)\varrho_1^{-1/N'}(x_1)\big]^{-N}, \tag{2.5}$$

for all $t \in [0,1]$, and π-a.e. $(x_0,x_1) \in M \times M$. Here ϱ_t is the density of the push forward of π under the map $(x_0,x_1) \mapsto \gamma_t(x_0,x_1)$.

We recall the definition of the measure-contraction property.

A Markov kernel on M is a map $Q : M \times \mathcal{B}(M) \to [0,1]$ (where $\mathcal{B}(M)$ denotes the Borel σ-algebra of M) with the following properties:

(i) for each $x \in M$ the map $Q(x,\cdot) : \mathcal{B}(M) \to [0,1]$ is a probability measure on M;
(ii) for each $A \in \mathcal{B}(M)$ the function $Q(\cdot,A) : M \to [0,1]$ is m-measurable.

Definition 2.7 *(Measure-contraction property).* Let two numbers $K, N \in \mathbb{R}$ with $N \geqslant 1$ be given. We say that (M,d,m) satisfies the *measure-contraction property* $\mathsf{MCP}(K,N)$ if and only if for each $0 < t < 1$ there exists a Markov kernel Q_t from M^2 to M such that for m^2-a.e. $(x,y) \in M$ and for $Q_t(x,y;\cdot)$-a.e. z the point z is a t-intermediate point of x and y, and such that for m-a.e. $x \in M$ and for every measurable $B \subset M$,

$$\begin{aligned} \int_M \varsigma_{K,N}^{(t)}\big(d(x,y)\big)\,Q_t(x,y;B)\,m(dy) &\leqslant m(B), \\ \int_M \varsigma_{K,N}^{(1-t)}\big(d(x,y)\big)\,Q_t(y,x;B)\,m(dy) &\leqslant m(B). \end{aligned} \tag{2.6}$$

2.2. Disintegration of measures

Given a measurable space $(R, \mathscr{R})$ and a function $r : R \to S$, with S generic set, we can endow S with the *push forward σ-algebra $\mathscr{S}$* of $\mathscr{R}$:

$$Q \in \mathscr{S} \quad \Longleftrightarrow \quad r^{-1}(Q) \in \mathscr{R},$$

which could be also defined as the biggest σ-algebra on S such that r is measurable. Moreover given a measure space $(R, \mathscr{R}, \rho)$, the *push forward measure* η is then defined as $\eta := (r_\sharp \rho)$.

Consider a probability space $(R, \mathscr{R}, \rho)$ and its push forward measure space $(S, \mathscr{S}, \eta)$ induced by a map r. From the above definition the map r is clearly measurable and inverse measure preserving.

Definition 2.8. A *disintegration* of ρ *consistent with* r is a map $\rho : \mathscr{R} \times S \to [0, 1]$ such that

(1) $\rho_s(\cdot)$ is a probability measure on $(R, \mathscr{R})$ for all $s \in S$,
(2) $\rho_\cdot(B)$ is η-measurable for all $B \in \mathscr{R}$,

and satisfies for all $B \in \mathscr{R}$, $C \in \mathscr{S}$ the consistency condition

$$\rho\big(B \cap r^{-1}(C)\big) = \int_C \rho_s(B)\, \eta(ds).$$

A disintegration is *strongly consistent with respect to* r if for all s we have $\rho_s(r^{-1}(s)) = 1$.

The measures ρ_s are called *conditional probabilities*.

We say that a σ-algebra $\mathcal{H}$ is *essentially countably generated* with respect to a measure m if there exists a countably generated σ-algebra $\hat{\mathcal{H}}$ such that for all $A \in \mathcal{H}$ there exists $\hat{A} \in \hat{\mathcal{H}}$ such that $m(A \vartriangle \hat{A}) = 0$.

We recall the following version of the disintegration theorem. See [2] for a direct proof.

Theorem 2.9 *(Disintegration of measures). Assume that $(R, \mathscr{R}, \rho)$ is a countably generated probability space, $R = \bigcup_{s \in S} R_s$ a partition of R, $r : R \to S$ the quotient map and $(S, \mathscr{S}, \eta)$ the quotient measure space. Then $\mathscr{S}$ is essentially countably generated w.r.t. η and there exists a unique disintegration $s \mapsto \rho_s$ in the following sense: if ρ_1, ρ_2 are two consistent disintegrations then $\rho_{1,s}(\cdot) = \rho_{2,s}(\cdot)$ for η-a.e. s.*

If $\{S_n\}_{n \in \mathbb{N}}$ is a family essentially generating $\mathscr{S}$ define the equivalence relation:

$$s \sim s' \quad \Longleftrightarrow \quad \big\{ s \in S_n \Longleftrightarrow s' \in S_n,\ \forall n \in \mathbb{N} \big\}.$$

Denoting with p the quotient map associated to the above equivalence relation and with $(L, \mathscr{L}, \lambda)$ the quotient measure space, the following properties hold:

- *$R_l := \bigcup_{s \in p^{-1}(l)} R_s = (p \circ r)^{-1}(l)$ is ρ-measurable and $R = \bigcup_{l \in L} R_l$;*
- *the disintegration $\rho = \int_L \rho_l\, \lambda(dl)$ satisfies $\rho_l(R_l) = 1$, for λ-a.e. l. In particular there exists a strongly consistent disintegration w.r.t. $p \circ r$;*
- *the disintegration $\rho = \int_S \rho_s\, \eta(ds)$ satisfies $\rho_s = \rho_{p(s)}$ for η-a.e. s.*

In particular we will use the following corollary.

Corollary 2.10. *If $(S, \mathscr{S}) = (X, \mathcal{B}(X))$ with X Polish space, then the disintegration is strongly consistent.*

3. Polar coordinates

From now on we will assume (M, d, m) to be a non-branching metric measure space satisfying $\mathsf{CD}_{loc}(K, N)$ for some $K, N \in \mathbb{R}$ and $N \geqslant 1$. Since we want to prove that (M, d, m) satisfies $\mathsf{MCP}(K, N)$ we also fix once forever $o \in M$.

Decompose $M = \bigcup_{r \geqslant 0} M_r$ with $M_r := \partial B_r(o)$ and, accordingly to this decomposition, m can be disintegrated in the following way

$$m = \int \bar{m}_r \, q(dr), \qquad q(A) = m\big(\big\{x\colon\ d(x, 0) \in A\big\}\big).$$

It is fairly easy to prove that the disintegration is strongly consistent. Indeed restrict m to $B_R(o)$, with $R > 0$, and consider any constant speed geodesic γ going from o to M_R and take $[0, R]$ as the quotient set. It follows that the quotient space is a Polish space and then by Corollary 2.10 the disintegration is strongly consistent. Letting $R \nearrow +\infty$, we obtain the strong consistency for the whole measure and q will be a locally finite measure, therefore:

$$\bar{m}_r\big(\big\{x\colon\ d(x, o) = r\big\}\big) = 1, \quad \text{for } q\text{-a.e. } r \in [0, \bar{R}].$$

Proposition 3.1. *The quotient measure $q \ll \mathcal{L}^1$.*

Proof. Since (M, d, m) satisfies $\mathsf{CD}_{loc}(K, N)$, from [1] (M, d, m) verifies $\mathsf{CD}^*(K, N)$, then defining

$$v(r) := m\big(\bar{B}_r(o)\big), \qquad s(r) := \limsup_{\delta \to 0} \frac{1}{\delta} m\big(\bar{B}_{r+\delta}(o) \setminus B_r(o)\big),$$

the map $r \mapsto v(r)$ is locally Lipschitz with s as weak derivative, Theorem 2.3 of [8]. Being s the density of q w.r.t. $\mathcal{L}^1$, it follows that $q \ll \mathcal{L}^1$. $\square$

With a slight abuse of notation $q(dr) = q(r)\mathcal{L}^1$. Let $m_r := q(r)\bar{m}_r$ so we have

$$m = \int m_r \, dr.$$

Let $s_r := m_r(M) = m_r(M_r) = \frac{d^+}{dr} m(B_r(o))$ and note that reduced Bishop–Gromov inequality, see [1], implies that for all $0 < r \leqslant R \leqslant \pi\sqrt{(N-1)/K^*}$

$$\frac{s_r}{s_R} \geqslant \left(\frac{\sin(r\sqrt{K^*/(N-1)})}{\sin(R\sqrt{K^*/(N-1)})}\right)^N, \tag{3.1}$$

where $K^* = K(N-1)/N$.

Fix $R > 0$ with $s_R > 0$ and let $(p_r)_{r \in [0,R]}$ denote the geodesic in $\mathcal{P}(M)$ connecting the probability measures $p_0 = \delta_{x_0}$ and $p_R = \frac{1}{s_R} m_R$. Note that for each r the measure p_r is supported on M_r. The next lemma follows straightforwardly from (3.1).

Lemma 3.2. *The measure p_r is absolute continuous with respect to the surface measure m_r.*

Let $\hat{h}_r(x) := \frac{dp_r}{dm_r}(x)$ denote the density. Clearly $\hat{h}_r$ can be defined arbitrarily outside M_r. Therefore for $\mathcal{L}^1$-a.e. $p_r = \hat{h}_r m_r$.

Remark 3.3. Let us consider the set of geodesic

$$\mathcal{G}_{[0,R]}(M) := \{\gamma : [0, R] \to M, \text{ constant speed geodesic}\}.$$

Let $\nu \in \mathcal{P}(\mathcal{G}_{[0,R]}(M))$ such that for $\mathcal{L}^1$-a.e. $r \in [0, R]$, $e_{r\sharp}\nu = p_r$. Neglecting a set of arbitrarily small ν-measure, we assume w.l.o.g. that

$$G := \operatorname{supp}[\nu] \subset \mathcal{G}_{[0,R]}(M), \qquad \hat{G}_r := e_r(G) \subset M_r, \qquad \hat{G} := \bigcup_{r \in [0,R]} \hat{G}_r \subset M,$$

with G compact and the maps $e_r : G \to \hat{G}_r$ and

$$\begin{aligned} e : (0, R) \times G &\to \hat{G}, \\ (r, \gamma) &\mapsto e_r(\gamma) := \gamma_r \end{aligned}$$

are both homeomorphisms. We also prefer to think of $\hat{h}_r$ as a function defined on G rather than on $\hat{G}$, hence define $h_r : G \to [0, \infty]$ by $h_r(\gamma) := \hat{h}_r(\gamma_r)$.

4. The $(N-1)$-dimensional estimate

Consider $H \subset G$, ν-measurable with $\nu(H) > 0$ and numbers $R_0, L_0, R_1, L_1 > 0$ with $R_0 < R_1$ such that $R_t + L_t < R$ for all $t \in [0, 1]$ where $R_t := (1-t)R_0 + tR_1$ and $L_t := (1-t)L_0 + tL_1$, then the following holds.

Lemma 4.1. *The curve*

$$t \mapsto \mu_t := \frac{1}{L_t \nu(H)} \int_0^R 1_{(R_t, R_t + L_t) \times H}\big(e^{-1}(x)\big)\, p_r(dx)\, \mathcal{L}^1(dr) \in \mathcal{P}(M) \tag{4.1}$$

is a geodesic.

Proof. Observe that coupling each $\gamma_{R_s + \lambda L_s}$ with $\gamma_{R_t + \lambda L_t}$ for $\lambda \in [0, 1]$, $\gamma \in H$ we obtain a d^2-cyclically monotone coupling of μ_s with μ_t. The property then follows straightforwardly. $\square$

Hence, the optimal transport is achieved by not changing the "angular" parts and coupling radial parts according to optimal coupling on $\mathbb{R}$. Observe that for each $t \in [0,1]$ the density $\varrho_t(x)$ of μ_t w.r.t. m is given by

$$\varrho_t(\gamma_r) = \begin{cases} \frac{1}{L_t \nu(H)} h_r(\gamma), & (r,\gamma) \in [R_t, R_t + L_t] \times H, \\ 0, & \text{otherwise}. \end{cases} \tag{4.2}$$

The following regularity result for densities holds true.

Lemma 4.2. *For ν-a.e. $\gamma \in G$, the function $r \mapsto h_r^{-1/N}(\gamma)$ is semi-concave on $(0, R)$ and satisfies in distributional sense*

$$\partial_r^2 h_r^{-1/N}(\gamma) \leqslant -\frac{K}{N} h_r^{-1/N}(\gamma).$$

Proof. Recall that $\mathsf{CD}_{loc}(K,N)$ implies $\mathsf{CD}^*(K,N)$. Consider the geodesic μ_t defined in (4.1) with $L_0 = L_1 = 1$ and apply the definition of $\mathsf{CD}^*(K,N)$ to get

$$h_s^{-1/N}(\gamma) \geqslant \frac{\sin(t-s)\sqrt{K/N}}{\sin(t-r)\sqrt{K/N}} h_r^{-1/N}(\gamma) + \frac{\sin(s-r)\sqrt{K/N}}{\sin(t-r)\sqrt{K/N}} h_t^{-1/N}(\gamma), \tag{4.3}$$

for all $0 < r < s < t < R$ and ν-a.e. $\gamma \in G$. The claim is equivalent to (4.3). $\square$

Now fix an open set $H \subset G$ and $[a,b] \subset [0,R]$ such that the curvature-dimension condition $\mathsf{CD}(K,N)$ holds true for all measures μ_0, μ_1 supported in $e([a,b] \times \bar{H})$. For each $R_0, R_1 \in (a,b)$ choose L_0, L_1 such that $R_0 + L_0, R_1 + L_1 \leqslant b$ and define $(\mu_t)_{t\in[0,1]}$ as before in (4.1). Moreover we have to consider the following map $\Phi : \mathcal{G}_{[0,R]}(M) \times [0,1] \to \mathcal{G}_{[0,1]}(M)$ with $\Phi(\gamma, s)$ being the geodesic $t \mapsto \eta_t = \gamma_{(1-t)(R_0+sL_0)+t(R_1+sL_1)}$. Consider

$$\tilde{\nu} := \Phi_\sharp\left(\frac{1}{\nu(H)} \nu \llcorner_H \otimes \mathcal{L}^1 \llcorner_{[0,1]}\right),$$

then $\mu_t = e_{t\sharp}\tilde{\nu}$.

Theorem 4.3. *For ν-a.e. $\gamma \in H$ and for sufficiently close $R_0 < R_1$ the following holds true*:

$$h_{R_{1/2}}^{-\frac{1}{N-1}}(\gamma) \geqslant \sigma_{K,N-1}^{(1/2)}(R_1 - R_0)\left\{h_{R_0}^{-\frac{1}{N-1}}(\gamma) + h_{R_1}^{-\frac{1}{N-1}}(\gamma)\right\}. \tag{4.4}$$

Proof. Consider the measures μ_0 and μ_1, the corresponding measure on the space of geodesics $\tilde{\nu}$ and recall that $\mu_t = \varrho_t m$.

Step 1. Condition $\mathsf{CD}_{loc}(K,N)$ for $t = 1/2$ and the assumptions on R_0, L_0 and R_1, L_1 imply that for $\tilde{\nu}$-a.e. $\eta \in \mathcal{G}_{[0,1]}(M)$

$$\varrho_{1/2}^{-1/N}(\eta_{1/2}) \geqslant \tau_{K,N}^{(1/2)}\big(d(\eta_0,\eta_1)\big)\left\{\varrho_0^{-1/N}(\eta_0) + \varrho_1^{-1/N}(\eta_1)\right\},$$

that can be formulated also in the following way: for $\mathcal{L}^1$-a.e. $s \in [0,1]$ and ν-a.e. $\gamma \in H$

$$\varrho_{1/2}^{-1/N}(\gamma_{R_{1/2}+sL_{1/2}}) \geqslant \tau_{K,N}^{(1/2)}\big(R_1 - R_0 + s|L_1 - L_0|\big)\left\{\varrho_0^{-1/N}(\gamma_{R_0+sL_0}) + \varrho_1^{-1/N}(\gamma_{R_1+sL_1})\right\}.$$

Then using (4.2) and the continuity of $r \mapsto h_r(\gamma)$ (Lemma 4.2), letting $s \searrow 0$, it follows that

$$(L_0+L_1)^{1/N} h_{R_{1/2}}^{-1/N}(\gamma) \geqslant \sigma_{K,N}^{(1/2)}(R_1-R_0)^{\frac{N-1}{N}} \left\{L_0^{1/N} h_{R_0}^{-1/N}(\gamma) + L_1^{1/N} h_{R_1}^{-1/N}(\gamma)\right\} \tag{4.5}$$

for all $R_0 < R_1 \in (a,b)$, all sufficiently small L_0, L_1 and ν-a.e. $\gamma \in H$, with exceptional set depending on R_0, R_1, L_0, L_1.

Step 2. Note that all the involved quantities in (4.5) are continuous w.r.t. R_0, R_1, L_0, L_1, therefore there exists a common exceptional set $H' \subset H$ of zero ν-measure such that (4.5) holds true for all $R_0 < R_1 \in (a,b)$, all sufficiently small L_0, L_1 and all $\gamma \in H \setminus H'$.

For fixed $R_0 < R_1 \in (a,b)$ and fixed $\gamma \in H \setminus H'$, varying L_0, L_1 in (4.5) yields

$$h_{R_{1/2}}^{-\frac{1}{N-1}}(\gamma) \geqslant \sigma_{K,N-1}^{(1/2)}(R_1-R_0)\left\{h_{R_0}^{-\frac{1}{N-1}}(\gamma) + h_{R_1}^{-\frac{1}{N-1}}(\gamma)\right\}.$$

Indeed the optimal choice is

$$L_0 = L\frac{h_{R_0}^{-1/(N-1)}(\gamma)}{h_{R_0}^{-1/(N-1)}(\gamma) + h_{R_1}^{-1/(N-1)}(\gamma)}, \qquad L_1 = L\frac{h_{R_1}^{-1/(N-1)}(\gamma)}{h_{R_0}^{-1/(N-1)}(\gamma) + h_{R_1}^{-1/(N-1)}(\gamma)}$$

for sufficiently small $L > 0$. □

5. The global estimates

From Theorem 4.3 we have that for every fixed $\gamma \in G \setminus H'$: for every $0 < R_0 < R$ there exists $\varepsilon > 0$ such that for all $R_0 < R_1 < R_0 + \varepsilon$ it holds

$$h_{R_{1/2}}^{-\frac{1}{N-1}}(\gamma) \geqslant \sigma_{K,N-1}^{(1/2)}(R_1-R_0)\left\{h_{R_0}^{-\frac{1}{N-1}}(\gamma) + h_{R_1}^{-\frac{1}{N-1}}(\gamma)\right\}.$$

We prove that midpoints inequality is equivalent to the complete inequality.

Lemma 5.1 *(Midpoints). Inequality* (4.4) *holds true if and only if*

$$h_{R_t}^{-\frac{1}{N-1}}(\gamma) \geqslant \sigma_{K,N-1}^{(1-t)}(R_1-R_0) h_{R_0}^{-\frac{1}{N-1}}(\gamma) + \sigma_{K,N-1}^{(t)}(R_1-R_0) h_{R_1}^{-\frac{1}{N-1}}(\gamma) \tag{5.1}$$

for all $t \in [0,1]$.

Proof. We only consider the case $K > 0$. The general case requires analogous calculations. Fix $0 \leqslant R_0 \leqslant R_1 \leqslant R$, put $\theta := R_1 - R_0$ and $h(s) := h_s(\gamma) = h(\gamma(s))$.

Step 1. For every $k \in \mathbb{N}$ we have

$$\begin{aligned} h^{-\frac{1}{N-1}}\left(R_0 + l2^{-k}\theta\right) \geqslant{}& \sigma_{K,N-1}^{(1/2)}\left(2^{-k+1}\theta\right) h^{-\frac{1}{N-1}}\left(R_0 + (l-1)2^{-k}\theta\right) \\ &+ \sigma_{K,N-1}^{(1/2)}\left(2^{-k+1}\theta\right) h^{-\frac{1}{N-1}}\left(R_0 + (l+1)2^{-k}\theta\right), \end{aligned}$$

for every odd $l = 0, \ldots, 2^k$.

Step 2. We perform an induction argument on k: suppose that inequality (5.1) is satisfied for all $t = l2^{-k+1} \in [0,1]$ with l odd, then (5.1) is verified by every $t = l2^{-k} \in [0,1]$ with l odd:

$$\begin{aligned}
&h^{-\frac{1}{N-1}}\big(R_0 + l2^{-k}\theta\big)\\
&\quad\geqslant \sigma_{K,N-1}^{(1/2)}\big(2^{-k+1}\theta\big)h^{-\frac{1}{N-1}}\big(R_0 + (l-1)2^{-k}\theta\big)\\
&\qquad + \sigma_{K,N-1}^{(1/2)}\big(2^{-k+1}\theta\big)h^{-\frac{1}{N-1}}\big(R_0 + (l+1)2^{-k}\theta\big)\\
&\quad\geqslant \sigma_{K,N-1}^{(1/2)}\big(2^{-k+1}\theta\big)\big[h^{-\frac{1}{N-1}}(R_0)\sigma_{K,N-1}^{(1-(l-1)2^{-k})}(\theta) + h^{-\frac{1}{N-1}}(R_1)\sigma_{K,N-1}^{((l-1)2^{-k})}(\theta)\big]\\
&\qquad + \sigma_{K,N-1}^{(1/2)}\big(2^{-k+1}\theta\big)\big[h^{-\frac{1}{N-1}}(R_0)\sigma_{K,N-1}^{(1-(l+1)2^{-k})}(\theta) + h^{-\frac{1}{N-1}}(R_1)\sigma_{K,N-1}^{((l+1)2^{-k})}(\theta)\big].
\end{aligned}$$

Following the calculation of the proof of Proposition 2.10 of [1], one obtains that

$$h^{-\frac{1}{N-1}}\big(R_0 + l2^{-k}\theta\big) \geqslant \sigma_{K,N-1}^{(1-l2^{-k})}(\theta)h^{-\frac{1}{N-1}}(R_0) + \sigma_{K,N-1}^{(l2^{-k})}(\theta)h^{-\frac{1}{N-1}}(R_1).$$

The claim is easily proved by the continuity of h and σ. □

We prove that (5.1) satisfies a local-to-global property.

Theorem 5.2 *(Local-to-global). Suppose that for every $R_0 \in [0,R]$ there exists $\varepsilon > 0$ such that whenever $R_0 < R_1 < R_0 + \varepsilon$ then* (5.1) *holds true for all $t \in [0,1]$. Then* (5.1) *holds true for all $0 < R_0 < R_1 \leqslant R$ and $t \in [0,1]$.*

Proof. We only consider the case $K > 0$. The general case requires analogous calculations. Fix $0 < R_0 < R_1 \leqslant R$, $\theta := R_1 - R_0$ and $h(s) := h_s(\gamma) = h(\gamma(s))$.

Step 1. According to our assumption, every point $R_0 \in [0,R]$ has a neighborhood $(R_0 - \varepsilon(R_0), R + \varepsilon(R_0))$ such that if R_1 belongs to that neighborhood then (5.1) is verified. By compactness of $[0,R]$ there exist $x_1,\ldots,x_n$ such that the family $\{B_{\varepsilon(x_i)/2}(x_i)\}_{i=1,\ldots,n}$ is a covering of $[0,R]$. Let $\lambda := \min\{\varepsilon(x_i)/2\colon\ i = 1,\ldots,n\}$. Possibly taking a lower value for λ, we assume that $\lambda = 2^{-k}\theta$. Hence we have

$$\begin{aligned}
h^{-\frac{1}{N-1}}\left(R_0 + \frac{1}{2}\theta\right) &\geqslant \sigma_{K,N-1}^{(1/2)}\big(2^{-k+1}\theta\big)h^{-\frac{1}{N-1}}\left(R_0 + \frac{1}{2}\theta - 2^{-k}\theta\right)\\
&\quad + \sigma_{K,N-1}^{(1/2)}\big(2^{-k+1}\theta\big)h^{-\frac{1}{N-1}}\left(R_0 + \frac{1}{2}\theta + 2^{-k}\theta\right).
\end{aligned}$$

Step 2. We iterate the above inequality:

$$\begin{aligned}
h^{-\frac{1}{N-1}}\left(R_0 + \frac{1}{2}\theta\right) &\geqslant \sigma_{K,N-1}^{(1/2)}\big(2^{-k+1}\theta\big)h^{-\frac{1}{N-1}}\left(R_0 + \frac{1}{2}\theta - 2^{-k}\theta\right)\\
&\quad + \sigma_{K,N-1}^{(1/2)}\big(2^{-k+1}\theta\big)h^{-\frac{1}{N-1}}\left(R_0 + \frac{1}{2}\theta + 2^{-k}\theta\right)\\
&\geqslant \sigma_{K,N-1}^{(1/2)}\big(2^{-k+1}\theta\big)\left[\sigma_{K,N-1}^{(1/2)}\big(2^{-k+1}\theta\big)h^{-\frac{1}{N-1}}\left(R_0 + \frac{1}{2}\theta - 2^{-k+1}\theta\right)\right.
\end{aligned}$$

$$
\begin{aligned}
&+\sigma_{K,N-1}^{(1/2)}\big(2^{-k+1}\theta\big)h^{-\frac{1}{N-1}}\left(R_0+\frac{1}{2}\theta\right)\Bigg]\\
&+\sigma_{K,N-1}^{(1/2)}\big(2^{-k+1}\theta\big)\Bigg[\sigma_{K,N-1}^{(1/2)}\big(2^{-k+1}\theta\big)h^{-\frac{1}{N-1}}\left(R_0+\frac{1}{2}\theta\right)\\
&+\sigma_{K,N-1}^{(1/2)}\big(2^{-k+1}\theta\big)h^{-\frac{1}{N-1}}\left(R_0+\frac{1}{2}\theta+2^{-k+1}\theta\right)\Bigg]\\
\geqslant{}&\sigma_{K,N-1}^{(1/2)}\big(2^{-k+1}\theta\big)^2h^{-\frac{1}{N-1}}\left(R_0+\frac{1}{2}\theta-2^{-k+1}\theta\right)\\
&+\sigma_{K,N-1}^{(1/2)}\big(2^{-k+1}\theta\big)^2h^{-\frac{1}{N-1}}\left(R_0+\frac{1}{2}\theta+2^{-k+1}\theta\right).
\end{aligned}
$$

Observing that $\sigma_{K,N-1}^{(1/2)}(\alpha)^2 \geqslant \sigma_{K,N-1}^{(1/2)}(2\alpha)$, it is fairly easy to obtain

$$
\begin{aligned}
h^{-\frac{1}{N-1}}\left(R_0+\frac{1}{2}\theta\right)\geqslant{}&\sigma_{K,N-1}^{(1/2)}\big(2^{-k+i+1}\theta\big)h^{-\frac{1}{N-1}}\left(R_0+\frac{1}{2}\theta-2^{-k+i}\theta\right)\\
&+\sigma_{K,N-1}^{(1/2)}\big(2^{-k+i+1}\theta\big)h^{-\frac{1}{N-1}}\left(R_0+\frac{1}{2}\theta+2^{-k+i}\theta\right),
\end{aligned}
$$

for every $i=0,\dots,k$. For $i=k-1$ Lemma 5.1 implies the claim. □

6. From local $\mathsf{CD}(K,N)$ to $\mathsf{MCP}(K,N)$

So we have proved that for any $0<R_0<R_1<R$ the density h_r, of p_r w.r.t. m_r, satisfies the following inequality:

$$
h_{R_t}^{-\frac{1}{N-1}}(\gamma)\geqslant\sigma_{K,N-1}^{(1-t)}(R_1-R_0)h_{R_0}^{-\frac{1}{N-1}}(\gamma)+\sigma_{K,N-1}^{(t)}(R_1-R_0)h_{R_1}^{-\frac{1}{N-1}}(\gamma) \tag{6.1}
$$

for ν-a.e. $\gamma\in G$ and all $t\in[0,1]$.

Consider $0<r_0<r_1\leqslant R$ and the following probability measure

$$
\mu_0:=\frac{1}{r_1-r_0}\int\limits_{(r_0,r_1)}\frac{m_r}{s_r}\,dr.
$$

Let $[0,1]\ni t\mapsto\mu_t\in\mathcal{P}_2(M,d,m)$ be the geodesic connecting μ_0 to $\mu_1=\delta_{x_0}$ with $\mu_t=\varrho_t m$. Let moreover $\pi_t\in\Pi(\mu_0,\mu_t)$ the corresponding optimal coupling.

Proposition 6.1. *Fix $t\in[0,1)$. Then for π_t-a.e. $(z_0,z_1)\in M^2$ the following holds true*

$$
\varrho_{ts}\big(\gamma_s(z_0,z_1)\big)^{-1/N}\geqslant\varrho_0(z_0)^{-1/N}\tau_{K,N}^{(1-s)}\big(d(z_0,z_1)\big)+\varrho_t(z_1)^{-1/N}\tau_{K,N}^{(s)}\big(d(z_0,z_1)\big), \tag{6.2}
$$

for every $s\in[0,1]$, where $\gamma_s(z_0,z_1)$ is the s-intermediate point on the geodesic γ connecting z_0 to z_1.

Proof. We use the following notation: for a given R consider the geodesic $(p_{R,r})_{s\in[0,R]}$ with $p_{R,0}=\delta_{x_o}$ and $p_{R,R}=m_R$. The same rule will apply to densities $h_{R,r}$.

Let $[0,1]\ni s\mapsto \Gamma_{st}:=\mu_{st}$ and observe that

$$\Gamma_s=\mu_{st}=\frac{1}{(1-st)(r_1-r_0)}\int\limits_{(1-st)(r_0,r_1)} h_{r/(1-t),r}m_r\,dr. \tag{6.3}$$

Consider $x_0\in M_{\bar r}$ with $r_0\leqslant\bar r\leqslant r_1$. Then the unique x_1 such that (x_0,x_1) is in the support of the optimal plan π_t, belongs to $M_{(1-t)\bar r}$. Then from Theorem 4.3 and (6.3)

$$\begin{aligned}
\big((r_1-r_0)\varrho_{st}\big(\gamma_s(x_0,x_1)\big)\big)^{-1/N}&=\left(\frac{1}{1-st}h_{\bar r,(1-st)\bar r}(\gamma)\right)^{-1/N}\\
&=\left(\frac{1}{(1-t)s+1-s}\right)^{-\frac{1}{N}}\left(h^{-\frac{1}{N-1}}_{\bar r,(1-t)s\bar r+(1-s)\bar r}(\gamma)\right)^{\frac{N-1}{N}}\\
&\geqslant(1-s)^{1/N}\big(\sigma^{(1-s)}_{K,N-1}(t\bar r)h^{-\frac{1}{N-1}}_{\bar r,\bar r}(\gamma)\big)^{\frac{N-1}{N}}\\
&\quad+\big((1-t)s\big)^{1/N}\big(\sigma^{(s)}_{K,N-1}(t\bar r)h^{-\frac{1}{N-1}}_{\bar r,(1-t)\bar r}(\gamma)\big)^{\frac{N-1}{N}}\\
&=\tau^{(1-s)}_{K,N}\big(d(z_0,z_1)\big)\big((r_1-r_0)\varrho_0(x_0)\big)^{-1/N}\\
&\quad+\tau^{(s)}_{K,N}\big(d(z_0,z_1)\big)\big((r_1-r_0)\varrho_t(z_1)\big)^{-1/N}.
\end{aligned}$$

The claim follows. □

So far we have proven that given $\mu_0:=m(A)^{-1}m\llcorner_A$, $x_0\in\operatorname{supp}[m]$ and the unique geodesic $[0,1]\ni t\mapsto\Gamma(t)$ such that $\Gamma(0)=\mu_0$, $\Gamma(1)=\delta_{x_0}$ and $\Gamma(t)=\varrho_t m$ for $t\in[0,1)$ we have for any $t\in[0,1)$:

$$\begin{aligned}
\mathcal{S}_{N'}\big(\Gamma(ts)|m\big)\leqslant-\int\limits_{M\times M}&\big[\tau^{(1-s)}_{K,N'}\big(d(x_0,x_1)\big)\varrho_0^{-1/N'}(x_0)\\
&+\tau^{(s)}_{K,N'}\big(d(x_0,x_1)\big)\varrho_t^{-1/N'}(x_1)\big]\,\pi_t(dx_0\,dx_1),
\end{aligned}$$

for all $s\in[0,1]$ and all $N'\geqslant N$, where $\pi_t=(P_0,P_t)_\sharp\Xi$.

We are ready to prove the main theorem of this section.

Theorem 6.2. *Let (M,d,m) be a non-branching metric measure spaces satisfying $\mathsf{CD}_{loc}(K,N)$. Then (M,d,m) satisfies $\mathsf{MCP}(K,N)$.*

Proof. *Step* 1. Let $\gamma:M^2\to\mathcal{G}(M)$ be the map introduced in Lemma 2.6 and define for each $t\in[0,1]$ a Markov kernel Q_t from M^2 to M by

$$Q_t(x,y;B):=1_B\big(\gamma_t(x,y)\big)$$

and for each pair t,x a measure $m_{t,x}=\int Q_t(x,y;\cdot)\,m(dy)$.

For each $x \in M$ let M_x denote the set of all $y \in M$ for which there exists a unique geodesic connecting x and y and let M_0 be the set of x such that $m(M \setminus M_x) = 0$. By assumption $m(M \setminus M_0) = 0$.

Step 2. Fix $x_0 \in M_0$ and $B \subset M$. Put $A_0 := \gamma_t(x_0, \cdot)^{-1}(B)$ and $\mu_0 := m(A_0)^{-1} m_{\llcorner A_0}$. Considering $s = 1$ in (6.2) it follows that

$$m(B)^{1/N} \geqslant \inf_{y \in A_0} \tau_{K,N}^{(t)}\big(d(y, x_0)\big) m(A_0)^{1/N},$$

or equivalently

$$m(B) \geqslant \inf_{y \in \gamma_t(x_0,\cdot)^{-1}(B)} \varsigma_{K,N}^{(t)}\big(d(y, x_0)\big) m\big(\gamma_t(x_0, \cdot)^{-1}(B)\big) = \inf_{z \in B} \varsigma_{K,N}^{(t)}\left(\frac{d(z, x_0)}{t}\right) m_{t,x_0}(B).$$

Decomposing B into a disjoint union $\bigcup_i B_i$ with $B_i = B \cap (\bar{B}_{\varepsilon i}(x_0) \setminus \bar{B}_{\varepsilon(i-1)}(x_0))$, and applying the previous estimate to each of the B_i we obtain as $\varepsilon \to 0$

$$m(B) \geqslant \int_B \varsigma_{K,N}^{(t)}\left(\frac{d(z, x_0)}{t}\right) m_{t,x_0}(dz)$$

or equivalently

$$m(B) \geqslant \int_B \varsigma_{K,N}^{(t)}\big(d(z, x_0)\big) Q_t(x_0, y; B)\, m(dy). \qquad \square$$

7. Outlook

In the last part of this note we sketch the most general case we can address using the approach introduced so far. We start recalling the definition of d-transform: for $f : M \to \bar{\mathbb{R}}$ Borel measurable

$$f^d(y) := \inf_{x \in M} \frac{d^2(x, y)}{2} - f(x),$$

f^d is the d-transform of f. Accordingly, a map is d-concave if it can be written as the d-transform of another map.

The setting Let $A \subset M$ be a Borel set and define the map $\varphi_A(x) := d^2(A, x)/2$ where $d(A, x) := \inf\{d(z, x)\colon z \in A\}$. Clearly φ_A is d-concave, indeed if

$$\infty_A(x) := \begin{cases} 0, & x \in A, \\ +\infty, & x \notin A, \end{cases}$$

then $\varphi_A = \infty_A^d$.

Definition 7.1. Let $A \subset M$ be a closed set. The set A is *d-convex* if

$$(-\infty_A)^{dd} = -\infty_A. \tag{7.1}$$

Remark 7.2. In the Euclidean case, i.e. $\mathbb{R}^n$ equipped with Euclidean distance, Definition 7.1 is equivalent to the standard notion of convexity. Indeed for $f:\mathbb{R}^n \to \bar{\mathbb{R}}$ with $f > -\infty$ and not identically $+\infty$, consider the Legendre transform

$$f^*(y) := \sup_{x\in\mathbb{R}^n} \langle y, x\rangle - f(x).$$

Then it is well know that $f^{**} = f$ if and only if f is convex and l.s.c. (see for instance [6, Chapter 3]). Since

$$(-f)^{dd}(x) = \|x\|^2 - \big(f + \|\cdot\|^2\big)^{**}(x),$$

it is fairly easy to conclude that $(-\infty_A)^{dd} = -\infty_A$ is equivalent to convexity, provided A is a closed set.

We will prove the analogous of Proposition 6.1 only for those optimal transport plan having $(\varphi_A, -\infty_A)$ as Kantorovich potentials. Define the set

$$\begin{aligned} \Gamma_A &:= \big\{(x,y)\in M\times M:\ \varphi_A(x) - \infty_A(y) = d^2(x,y)\big\} \\ &= \left\{(x,y)\in M\times A:\ \varphi_A(x) = \frac{d^2(y,x)}{2}\right\} \end{aligned}$$

and the corresponding family of optimal dynamical transference plan

$$\boldsymbol{\gamma}_A := \big\{\boldsymbol{\gamma}\in\mathcal{P}\big(\mathcal{G}(M)\big):\ (e_0,e_1)_\sharp(\boldsymbol{\gamma})(\Gamma_A) = 1,\ e_{0\sharp}\boldsymbol{\gamma} = \varrho_0 m\big\}.$$

Theorem 7.3. *Let $A\subset M$ be compact and d-convex. Then every $\boldsymbol{\gamma}\in\boldsymbol{\gamma}_A$ satisfies the following: for every $t\in[0,1)$ we have $e_{t\sharp}\boldsymbol{\gamma} = \varrho_t m$ and*

$$\begin{aligned} \mathcal{S}_{N'}(e_{ts\sharp}\boldsymbol{\gamma}|m) \leqslant -\int\limits_{M\times M} &\big[\tau_{K,N'}^{(1-s)}\big(d(x_0,x_1)\big)\varrho_0^{-1/N'}(x_0) \\ &+ \tau_{K,N'}^{(s)}\big(d(x_0,x_1)\big)\varrho_t^{-1/N'}(x_1)\big]\,\pi_t(dx_0\,dx_1), \end{aligned}$$

for all $s\in[0,1]$ and all $N'\geqslant N$, where $\pi_t = (e_0,e_t)_\sharp\boldsymbol{\gamma}$.

We present an outline of the proof.

Proof. Due to non-branching assumption, $\mathsf{CD}(K,N)$ implies $\mathsf{CD}_{LV}(K,N)$, introduced by Lott and Villani in [3]. The latter implies that every geodesic consists of absolute continuous measures at intermediate times, whenever one of the two endpoints is absolute continuous (see [9, Theorem 30.19]).

The proof of this result is preserved if we replace all the coefficients $\tau_{K,N}$ by coefficients $\sigma_{K,N}$. The corresponding curvature-dimension condition $\mathsf{CD}^*_{LV}(K,N)$ follows from our condition $\mathsf{CD}_{LV}(K,N)$, due to the non-branching assumption. It follows therefore that

$$e_{t\sharp}\boldsymbol{\gamma} = \varrho_t m, \quad \forall t\in[0,1).$$

Note that $\mu_1(\partial A) = 1$. Therefore every geodesic belonging to the support of $\boldsymbol{\gamma}$ never enters inside A.

Polar coordinates Consider the following set

$$\Gamma_A(1) := \{\gamma_t:\ (\gamma, t) \in \operatorname{supp}(\boldsymbol{\gamma}) \times [0, 1)\}.$$

We only need a disintegration of m restricted to $\Gamma_A(1)$. Denote with

$$B_r(A) := \{x:\ d_A(x) \leqslant r\}.$$

Consider the family $\{\partial B_r(A)\}_{r>0}$ giving a partition of $\Gamma_A(1)$. It follows that

$$m_{\llcorner \Gamma_A(1)} = \int\limits_{(0,\infty)} \bar{m}_r\, q(dr), \quad \bar{m}_r\big(\partial B_r(A)\big) = 1.$$

Since the map $d(A, x)$ is Lipschitz and with strictly positive upper gradient on $\partial B_r(A)$ for $r > 0$, it follows from the coarea formula in metric measure spaces (see Proposition 4.2 of [4]) that

$$m_{\llcorner \Gamma_A(1)} = \int\limits_{(0,\infty)} \bar{m}_r q(r)\, \mathcal{L}^1(dr) = \int\limits_{(0,\infty)} m_r\, \mathcal{L}^1(dr). \tag{7.2}$$

Estimate in codimension 1 In the same way we disintegrate $\boldsymbol{\gamma}$:

$$\boldsymbol{\gamma} = \int \boldsymbol{\gamma}_r\, dr, \quad \|\boldsymbol{\gamma}_r\|^{-1}\boldsymbol{\gamma}_r\big(\{\gamma:\ d(A, \gamma_0) = r\}\big) = 1.$$

So fix R and consider the constant speed geodesic $(p_r)_{r\in[0,R]}$ such that $p_R = e_{0\sharp}\|\boldsymbol{\gamma}_R\|^{-1}\boldsymbol{\gamma}_R$ and $p_0 = e_{1\sharp}\|\boldsymbol{\gamma}_R\|^{-1}\boldsymbol{\gamma}_R$. Since $\mu_t \ll m$ for every $t \in [0, 1)$ it follows that

$$p_r = h_r m_r.$$

Now we can consider the family of geodesics (4.1) depending on R_i and L_i for $i = 1, 2$. The very same proof of Theorem 4.3 gives for sufficiently close $0 < R_0 < R_1$ the following:

$$h_{R_{1/2}}^{-\frac{1}{N-1}}(\gamma) \geqslant \sigma_{K,N-1}^{(1/2)}(R_1 - R_0)\big\{h_{R_0}^{-\frac{1}{N-1}}(\gamma) + h_{R_1}^{-\frac{1}{N-1}}(\gamma)\big\}.$$

As already shown during this note, the above estimates pass from local-to-global and therefore it holds true for any $0 < R_0 < R_1$.

Full-dimensional estimate We use the following notation: for a given R consider the geodesic $(p_{R,r})_{s\in[0,R]}$ with $p_{R,0} = e_{1\sharp}(\|\boldsymbol{\gamma}_R\|^{-1}\boldsymbol{\gamma}_R)$ and $p_{R,R} = e_{0\sharp}(\|\boldsymbol{\gamma}_R\|^{-1}\boldsymbol{\gamma}_R)$. Then for every $t \in [0, 1)$:

$$p_{R,(1-t)R} = \frac{\varrho_t}{\int \varrho_t m_{(1-t)R}} m_{(1-t)R}.$$

Since

$$\int_{\{R>0\}} m_{(1-t)R}\,\mathcal{L}^1(dR) = \frac{1}{1-t} m \llcorner_{\Gamma_A(1)},$$

we find the information on the density of the missing direction in the following way:

$$\mu_t = \int_{\{R>0\}} \frac{\varrho_t}{\int \varrho_t m_{(1-t)R}} m_{(1-t)R} \left(\int \varrho_t m_{(1-t)R} \right) dR = \frac{1}{(1-t)} \int_{\tau>0} \frac{\varrho_t}{\int \varrho_t m_\tau} \left(\int \varrho_t m_\tau \right) m_\tau \, d\tau.$$

It follows that the inverse of the 1-dimensional density evolves linearly with t. Imitating the proof of Proposition 6.1, for $t \in [0,1)$, for $(e_0, e_t)_\sharp \boldsymbol{\gamma}$-a.e. $(z_0, z_1) \in M^2$ the following holds true

$$\varrho_{ts}\big(\gamma_s(z_0,z_1)\big)^{-1/N} \geqslant \varrho_0(z_0)^{-1/N} \tau_{K,N}^{(1-s)}\big(d(z_0,z_1)\big) + \varrho_t(z_1)^{-1/N} \tau_{K,N}^{(s)}\big(d(z_0,z_1)\big),$$

for every $s \in [0,1]$, where $\gamma_s(z_0,z_1)$ is the s-intermediate point on the geodesic γ connecting z_0 to z_1. Integrating the previous inequality, we have the claim. □

To conclude this note we want to list the differences between the general globalization theorem, our case and the measure-contraction property. Assume that (M,d,m) satisfies $\mathsf{CD}_{loc}(K,N)$ and let μ_0 be an absolutely continuous probability measure and φ be a d-concave Kantorovich potential for a dynamical optimal transference plan:

MCP: prove (6.2) for every μ_0 and every $\varphi = (-\infty_{\{z\}})^d$, for every $z \in M$;
Note: prove (6.2) for every μ_0 and every $\varphi = (-\infty_A)^d$, with A d-convex;
CD: prove (6.2) for every μ_0 and every φ;

where by *Note* we mean the level of generality obtained in this paper. It is clear that μ_1 is determined by the choice of μ_0 and φ.

References

[1] K. Bacher, K.T. Sturm, Localization and tensorization properties of the curvature-dimension condition for metric measure spaces, J. Funct. Anal. 259 (1) (2010) 28–56.
[2] S. Bianchini, L. Caravenna, On the extremality, uniqueness and optimality of transference plans, Bull. Inst. Math. Acad. Sin. (N.S.) 4 (4) (2009) 353–454.
[3] J. Lott, C. Villani, Ricci curvature for metric-measure spaces via optimal transport, Ann. of Math. 169 (3) (2009) 903–991.
[4] M. Miranda Jr., Functions of bounded variations on "good" metric spaces, J. Math. Pures Appl. 82 (2003) 975–1004.
[5] S.-I. Ohta, On the measure contraction property of metric measure spaces, Comment. Math. Helv. 82 (2007) 805–828.
[6] S.T. Rachev, L. Ruchendorf, Mass Transportation Problems, vol. I: Theory, Springer, 1998.
[7] K.T. Sturm, On the geometry of metric measure spaces. I, Acta Math. 196 (1) (2006) 65–131.
[8] K.T. Sturm, On the geometry of metric measure spaces. II, Acta Math. 196 (1) (2006) 133–177.
[9] C. Villani, Optimal Transport, Old and New, Springer, 2008.